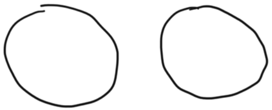
Links



A link is a set of knotted loops all tangled up together. Two links are considered to be the same if we can deform the one link to the other link without ever having any one of the loops intersect itself or any of the other loops in the process.

Links

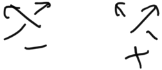
2-component link

3-component link

A knot will be considered a link of one component. if two projections represent the same link, there must be a sequence of Reidemeister moves to get from the one projection to the other.
A link is called splittable if the components of the link can be deformed so that they lie on different sides of a plane in three-space.

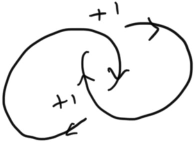
Linking Number

ORIENTED LINK



$$Lk(I,J) = \frac{1}{2}\big((-1)+(-1)\big) = -1$$

$Lk(I,J) = \frac{1}{2}\sum_{\text{all pairwise crossings}}$ sign of crossing $= \frac{1}{2}\sum_{k \in C} \epsilon_k$, where C is the set of pairwise crossings and $\epsilon_k = \pm 1$

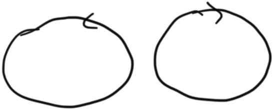


$$Lk(I,J) = 1$$

Linking Number

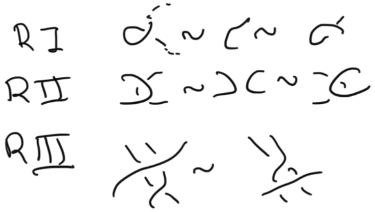
Hopf link

$\neq$



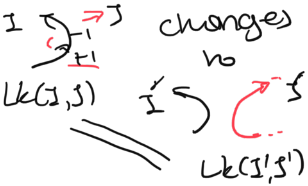
$Lk = -1$ $\qquad$ $Lk = 0$

$$Lk = \frac{1}{2}\left((-1)+\cancel{(+1)}+\cancel{(-1)}+(-1)\right)$$
$$= -1$$



RI involves a crossing of one component with itself it does not change Lk.

RII: Suppose the 2 arcs belong to different components



The linking number is an invariant.

We just need to prove it is invariant under Reidemeister moves.

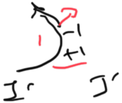
$$Lk(J',J') = \frac{1}{2} \sum_{\substack{\text{crossings} \\ \text{between } J',J'}} \varepsilon_c$$

$$= \frac{1}{2} \Big\{ \sum_{\text{crossings between } J,J} \varepsilon_c - \big[\underbrace{(-1)+(+1)}_{=0} \big] \Big] = Lk(J,J)$$

Linking Number

Similarly if we change

to

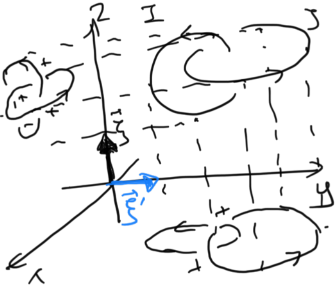


$$Lk(I',J') = \frac{1}{2}\sum_{\substack{\text{crossings}\\ \text{between } I',J'}} \varepsilon_c = \frac{1}{2}\left[\sum_{\substack{\text{cross.}\\ \text{between}\\ I,J}} + \overbrace{\left[(+1)+(-1)\right]}^{=0}\right] = \frac{1}{2}\sum_{\substack{\text{cross}\\ I,J}} \varepsilon_c = Lk(I,J)$$

Exercise: RIII

Linking Number

Since the linking nr is an invariant, it does not depend on the diagram.



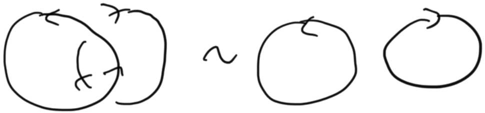
We project the link from 3-space to a plane with normal vector $\vec{\xi}$. (ie, on the xy-plane)
We compute $Lk(I,J) = \frac{1}{2}((+1)+(+1)) = 1$

Next: We project to the plane with normal vector $\vec{\xi}'$ (ie on the xz-plane). We compute:

$$Lk(I,J) = \frac{1}{2}((+1)+(+1)+(+1)+(-1)) = 1$$

Linking Number

Links



$Lk = 0$ ✓

A link is called Brunnian if the link itself is nontrivial, but the removal of any one of the components leaves us with a set of trivial unlinked circles. These links are named after Hermann Brunn, who drew pictures of such links back in 1892.

$Lk(I,J)=0$
$Lk(I,K)=0$
$Lk(J,K)=0$

$\not\sim$

Writhe of a knot diagram

$+$ $\qquad$ $-$

We define the **writhe of a diagram** of a knot to be:

$$\sum_{\substack{\text{crossings}\\ \text{in diagram}\\ \text{of knot}}} \varepsilon_c \text{, where } \varepsilon_c = \pm 1 \text{ (sign of crossing).}$$

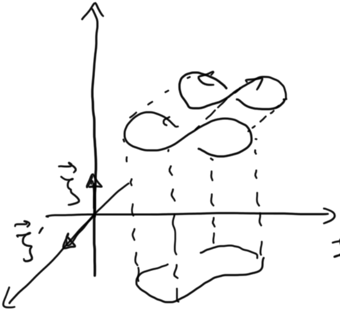
Writhe of diagram $= 0$

Writhe of diagram $= +1$

writhe of diagram $= -1$

So, the writhe of a diagram of a knot is not invariant under Reidemeister moves.

Writhe of a knot diagram



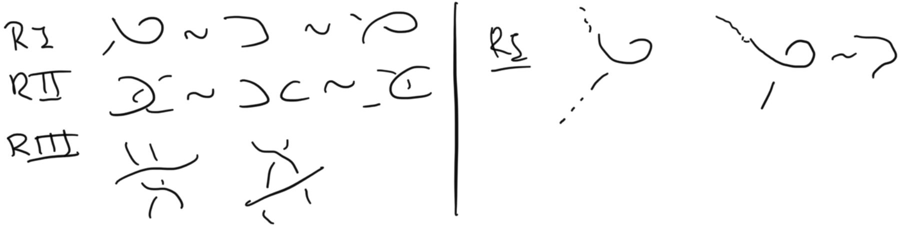
Let's denote $Wr_{\vec{\xi}}$ = the writhe of the diagram of the knot when we project the curve on the plane with normal vector $\vec{\xi}$

In this example: $Wr_{\vec{\xi}} = 0$, $Wr_{\vec{\xi}'} = +1$.

Tricolorability

We will say that a strand in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between. We will say that a projection of a knot or link is tricolorable if each of the strands in the projection can be colored one of three different colors, so that at each crossing, either three different colors come together or all the same color comes together. In order that a projection be tricolorable, we further require that at least two of the colors are used.

Tricolorability



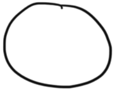
For our purposes, the most important fact is that if a projection of a knot is tricolorable, then the Reidemeister moves will preserve the tricolorability.

Tricolorability

not tricolorable

We have just shown there is at least one other knot besides the unknot. In fact, any knot that is tricolorable must be distinct from the unknot.

is tricolorable

So,  $\neq$ 

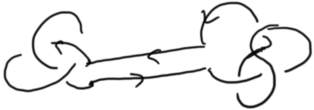
unknot

trefoil knot

Composition of knots

K_1

K_2



$K_1 \# K_2$

Let K_1, K_2 be two knots, then $K_1 \# K_2$ is obtained by removing an arc from each, connect the ends to get a single component taking into account orientation.

$\#$ is commutative

$\bigcirc$ is an identity element $K \# \bigcirc = K$

unknot

<u>but</u> there is no inverse

<u>Exercise</u>: Knots form a semigroup under $\#$

We call a knot a composite knot if it can be expressed as the composition of two knots, neither of which is the trivial knot. This is in analogy to the positive integers, where we call an integer composite if it is the product of positive integers, neither of which is equal to 1. The knots that make up the composite knot are called factor knots.

If a knot is not the composition of any two nontrivial knots, we call it a prime knot.

Example of a prime a knot