Applied Knot Theory

August 31, 2020

The normalized Kauffman bracket polynomial

$\langle \times \rangle = A\langle \asymp \rangle + B\langle)(\rangle$, $\langle K \rangle = \sum_{\sigma \in S} \underbrace{\langle K | S \rangle}_{\text{labels}} d^{\|S\|}$

$B = A^{-1}$, $d = -A^2 - A^{-2}$

$\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$, $\langle O \rangle = 1$, $\langle O\,K \rangle = (-A^2 - A^{-2})\langle K \rangle$

$\Rightarrow$ invariant under RII, III

under RI:

$\langle \gamma \rangle = -A^3 \langle - \rangle$ $(*)$

$\langle \gamma \rangle = -A^{-3} \langle - \rangle$ $(**)$

We define the normalized bracket polynomial
$f_K = (-A^3)^{-wr(K)}\langle K \rangle$

bracket pol.

We now use oriented diagrams.

We notice that
$$\left.\begin{aligned} wr(\gamma) &= 1 + wr(\rightarrow) \\ wr(\gamma) &= 1 + wr(\rightarrow) \end{aligned}\right\} (*)$$
$$\left.\begin{aligned} wr(\gamma) &= -1 + wr(\rightarrow) \\ wr(\gamma) &= -1 + wr(\rightarrow) \end{aligned}\right\} (**)$$

The normalized Kauffman bracket polynomial

The normalized bracket polynomial is invariant under ambient isotopy. It is a polynomial with positive and negative powers, these are called Laurent polynomials.

$wr(K)$ and $\langle K \rangle$ are invariant under RII, RIII

$f_K = (-A^3)^{-wr(K)} \langle K \rangle$ will also be invariant under RII, RIII

what about RI?

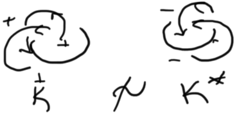
$\Rightarrow f_K(\gamma) = (-A^3)^{-wr(\gamma)} \langle \gamma \rangle =$

$= (-A^3)^{-(1+wr(\longrightarrow))} (-A^3) \langle \longrightarrow \rangle =$

$= (-A^3)^{-wr(\longrightarrow)} \langle \longrightarrow \rangle = f_K(\longrightarrow)$

The normalized Kauffman bracket polynomial of mirror images

$$f_K = (-A^3)^{-wr(K)} \langle K \rangle$$



Let K, K^* denote a knot and its mirror image.
Note that a crossing change in a diagram gives a switch of roles of A and A^{-1} in the polynomial.

$$\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$$
$$\langle \times \rangle = A^{-1} \langle \asymp \rangle + A \langle)(\rangle$$

So, we have $\langle K^* \rangle (A) = \langle K \rangle (A^{-1})$
and $f_{K^*}(A) = f_K(A^{-1})$.
Thus, if $f_K(A) \neq f_K(A^{-1})$, then K is not ambient isotopic to K^*.
So we can detect chirality!

$$f_{K^*}(A) = (-A^3)^{-wr(K^*)} \langle K^* \rangle (A) = (-A^3)^{wr(K)} \langle K \rangle (A^{-1})$$
$$= (-(A^{-1})^3)^{-wr(K)} \langle K \rangle (A^{-1})$$
$$= f_K(A^{-1})$$

So, it can distinguish the left and right handed Tr

$$\langle \; \rangle = -A^5 - A^{-3} + A^{-7}$$
$$f = (-A^3)^{-3} \langle \; \rangle = A^{-4} + A^{-12} - A^{-16}$$
$$f = (-A^3)^{-(-3)} \langle \; \rangle = A^4 + A^{12} - A^{16}$$

The Jones polynomial

$$V_K(t) = f_K(t^{-1/4}) \quad \text{Jones polynomial} \quad \boxed{A = t^{-1/4}}$$

The Jones polynomial $V_K(t)$ is a Laurent polynomial in t assigned to an oriented link K so that the following properties are satisfied:

(1) $V_K(t)$ is invariant under ambient isotopy

(2) $V_{\bigcirc}(t) = 1$

(3) $t^{-1} V_{\times_+} - t\, V_{\times_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) \cdot V_{\rightrightarrows}$

Pf. (1) $V_K(t) = f_K(t^{-1/4})$ which is invariant.

(2) $V_{\bigcirc}(t) = f_{\bigcirc}(t^{-1/4}) = (-(t^{-1/4})^3)^{-wr(\bigcirc)} \langle \bigcirc \rangle = 1$

(3)
$$\left.\begin{aligned} \langle \times \rangle &= A \langle \asymp \rangle + B \langle)(\rangle \\ \langle \times' \rangle &= B \langle \asymp \rangle + A \langle)(\rangle \end{aligned}\right\}$$

$$B^{-1} \langle \times \rangle - A^{-1} \langle \times' \rangle = \left(\frac{A}{B} - \frac{B}{A}\right) \langle \asymp \rangle$$

$$\boxed{A \langle \times \rangle - A^{-1} \langle \times' \rangle = (A^2 - A^{-2}) \langle \asymp \rangle}$$

$$\boxed{A\langle\ \rangle - A^{-1}\langle\ \rangle = (A^2 - A^{-2})\langle\ \rangle \quad (*)}$$

Let $\omega = wr(\ )$. Then $wr(\ ) = \omega + 1$

$wr(\ ) = \omega - 1$

Let $\alpha = -A^3$

Multiply $(*)$ by $\alpha^{-\omega} = (-A^3)^{-\omega}$

$$A\langle\ \rangle\alpha^{-\omega} - A^{-1}\langle\ \rangle\alpha^{-\omega} = (A^2 - A^{-2})\langle\ \rangle\alpha^{-\omega}$$

$$A\cdot\alpha\underbrace{\langle\ \rangle\alpha^{-(\omega+1)}}_{V_{\ }} - A^{-1}\alpha^{-1}\langle\ \rangle\alpha^{-(\omega-1)} = (A^2 - A^{-2})\langle\ \rangle\alpha^{-\omega}$$

$$-A^4\, V_{\ } \quad + A^{-4}\, V_{\ } = (A^2 - A^{-2})\, V_{\ }$$

for $A = t^{-1/4}$

$$t^{-1} V_{\ } - t\, V_{\ } = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{\ }$$

The Jones polynomial

$$V_K(t) = f_K(t^{-1/4})$$

Jones polynomial

⊗ The writhe of the diagram of K' does not depend on orientation

Let K be a link, let $K_1, \ldots, K_n$ be its components. Let K' be a link obtained from K by reversing the direction of one component. Let $\lambda = lk(K_1, K - K_1)$ be the total linking number of K_1 with the rest of K. Then $V_{K'}(t) = t^{-3\lambda} V_K(t)$.

Proof: The effect of reversing of direction of K_1 on the writhe of the link:

$lk(K_1, K-K_1) = \frac{1}{2} \sum_{c \in K_1 \cap K - K_1} \text{sign}(c)$ ← signed crossings between components

On a diagram: $w(K) = w(K - K_1) + w(K_1) + \sum_{c \in K_1 \cap K - K} \text{sign}(c)$

signed crossings of K_1 with itself

signed crossings other than K between K_1 and the rest.

$w(K') = w(K-K_1) + \boxed{w(K_1)} - \sum$ ⊗

$w(K') = w(K) - 2\sum$

$= w(K) - 4\, lk(K_1, K-K_1)$

$w(K') = w(K) - 4\lambda$. Since $\langle K' \rangle = \langle K \rangle$

We get $f_{K'}(A) = (-A^3)^{4\lambda} f_K(A) \rightsquigarrow A = t^{-1/4}$ gives the result.